目录

第六章　带电粒子与电磁场耦合

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前面的章节中我们考察了复标量场与电磁场的耦合系统。本章开始考察带电粒子与电磁场的耦合系统。对于电动力学在宏观世界的应用而言，这样的系统也许更为重要。因为宏观世界很少有机会出现一个复标量场来与电磁场相互作用，但却常常需要处理带电粒子与电磁场相互作用的问题。本章先考察自由粒子体系的能量动量张量，之后再进一步考察它们和电磁场的耦合。

6.1　多粒子系统的能量动量张量

考虑多个相对论性自由粒子所构成的系统，我们以$n=1,2,...,N$来标记不同的粒子。根据《经典力学新讲》第三章的知识可以知道，这个系统的作用量可以写成

$$S[x(s)] = -\sum_n m_n \int d\tau_n = -\sum_n m_n \int ds_n \sqrt{-\eta_{\mu\nu}\frac{dx_n^\mu}{ds_n}\frac{dx_n^\nu}{ds_n}}. \tag{6.1}$$

式中x_n^μ为粒子n的时空坐标，m_n为它的质量，τ_n为它的固有时，s_n为它的世界线参数。由于上式在$s_n \to \widetilde{s}_n(s_n)$的重参数化之下保持不变，所以$s_n$的选择有很大的任意性，特别的，我们可以将$s_n$选作固有时$\tau_n$。

不妨以固有时参数(即$s_n = \tau_n$)下的粒子作用量来进一步讨论。我们可以利用最小作用量原理求出上述自由粒子系统的运动微分方程，为此注意到

$$
\begin{aligned}
\delta S &= -\sum_n m_n \int \delta(d\tau_n) = -\sum_n m_n \int \frac{1}{2}\frac{\delta(d\tau_n^2)}{d\tau_n} \\
&= \sum_n m_n \int \frac{1}{2}\delta(\eta_{\mu\nu} dx_n^\mu dx_n^\nu)/d\tau_n \\
&= \sum_n m_n \int \frac{dx_n^\mu}{d\tau_n}\eta_{\mu\nu}\delta(dx_n^\nu) \\
&= \sum_n m_n \int \frac{dx_n^\mu}{d\tau_n}\eta_{\mu\nu} d(\delta x_n^\nu).
\end{aligned}
\tag{6.2}
$$

将最后一行的结果分部积分，并利用在边界上$\delta x_n^\mu = 0$, 即可得

$$
\delta S = -\sum_n m_n \int d\tau_n \frac{d^2 x_n^\mu}{d\tau_n^2}\eta_{\mu\nu}\delta x_n^\nu. \tag{6.3}
$$

根据最小作用量原理的$\delta S = 0$，即可得

$$
m_n \frac{d^2 x_n^\mu}{d\tau_n^2} = 0. \tag{6.4}
$$

定义第n个粒子的四维动量$p_n^\mu = (p_n^0, \mathbf{p}_n)$为

$$
p_n^\mu = m_n \frac{dx_n^\mu}{d\tau_n}. \tag{6.5}
$$

则上述运动微分方程可以重写成

$$
\frac{dp_n^\mu}{d\tau_n} = 0. \tag{6.6}
$$

另外，利用$x_n^0 = t_n$, $d\tau_n = \sqrt{1-\mathbf{v}_n^2}dt_n$，我们也可以得到

$$
p_n^0 = \frac{m_n}{\sqrt{1-\mathbf{v}_n^2}}, \quad \mathbf{p}_n = \frac{m_n \mathbf{v}_n}{\sqrt{1-\mathbf{v}_n^2}}. \tag{6.7}
$$

进而也有

$$
\mathbf{v}_n = \frac{\mathbf{p}_n}{p_n^0}. \tag{6.8}
$$

作用量(6.1)显然具有$x_n^\mu \to \widetilde{x}_n^\mu = x_n^\mu + a^\mu$($a^\mu$为常矢量) 的时空坐标平移不变性。为了考察这一时空平移对称性所对应的能动量张量，我们考虑如下局域化的无穷小时空坐标变换

$$x_n^\mu \to \widetilde{x}_n^\mu = x_n^\mu + \epsilon^\mu(x_n) \Leftrightarrow \delta x_n^\mu = \epsilon^\mu(x_n). \tag{6.9}$$

(从而$\delta(dx_n^\mu) = d(\delta x_n^\mu) = \frac{\partial \epsilon^\mu}{\partial x_n^\nu} dx_n^\nu$) 完全仿照(6.2)式，可得在此局域时空坐标变换下，作用量的改变量为

$$\begin{aligned}
\delta S &= \sum_n m_n \int \frac{dx_n^\mu}{d\tau_n} \eta_{\mu\nu} d(\delta x_n^\nu) \\
&= \sum_n m_n \int (\frac{dx_n^\mu}{d\tau_n}) \frac{\partial \epsilon_\mu}{\partial x_n^\nu} dx_n^\nu \\
&= \sum_n m_n \int d\tau_n (\frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n}) \frac{\partial \epsilon_\mu}{\partial x_n^\nu}.
\end{aligned} \tag{6.10}$$

通过引入四维时空δ函数$\delta^4\big(x - x_n(\tau_n)\big) = \prod_{\mu=0}^{3} \delta\big(x^\mu - x_n^\mu(\tau_n)\big)$, 可以进一步将上面结果写成

$$\begin{aligned}
\delta S &= \sum_n m_n \int d\tau_n (\frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n}) \frac{\partial \epsilon_\mu}{\partial x_n^\nu} \\
&= \int d^4x \Big[\sum_n m_n \int d\tau_n (\frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n}) \delta^4\big(x - x_n(\tau_n)\big) \Big] \partial_\nu \epsilon_\mu(x).
\end{aligned} \tag{6.11}$$

从而易知，与时空平移对称性对应的能动量张量为

$$T^{\mu\nu} = \sum_n m_n \int d\tau_n (\frac{dx_n^\mu}{d\tau_n} \frac{dx_n^\nu}{d\tau_n}) \delta^4\big(x - x_n(\tau_n)\big). \tag{6.12}$$

显然，这个能动量张量是一个对称张量。

为了看清楚上述能动量张量的意义，我们将它重写成，

$$T^{\mu\nu} = \sum_n m_n \int ds_n (\frac{dx_n^\mu}{ds_n} \frac{dx_n^\nu}{d\tau_n}) \delta^4\big(x - x_n(s_n)\big). \tag{6.13}$$

然后取世界线参数$s_n = x_n^0$, 则当我们做完对s_n的积分后，即有能动量密度

$$T^{0\mu} = \sum_n m_n (\frac{dx_n^\mu}{d\tau_n}) \delta^3\big(\mathbf{x} - \mathbf{x}_n(t)\big) = \sum_n p_n^\mu \delta^3\big(\mathbf{x} - \mathbf{x}_n(t)\big). \tag{6.14}$$

式中$\delta^3\big(\mathbf{x}-\mathbf{x}_n(t)\big)=\prod_{i=1}^3\delta\big(x^i-x_n^i(t)\big)$。从而可知与时空平移对称性对应的守恒量为

$$\int d^3\mathbf{x}T^{0\mu}=\sum_n p_n^\mu, \tag{6.15}$$

即粒子系统总的四维动量守恒。类似的，在取$s_n=x_n^0$并做完对s_n的积分以后，也有流密度

$$\begin{aligned}T^{i\mu}&=\sum_n m_n\big(\frac{dx_n^\mu}{d\tau_n}\frac{dx_n^i}{dt}\big)\delta^3\big(\mathbf{x}-\mathbf{x}_n(t)\big)\\&=\sum_n p_n^\mu v_n^i\delta^3\big(\mathbf{x}-\mathbf{x}_n(t)\big).\end{aligned} \tag{6.16}$$

综合(6.14)式和(6.16)式，可知粒子系统的能动量张量可以表达成

$$T^{\mu\nu}=\Big[\sum_n\frac{p_n^\mu p_n^\nu}{p_n^0}\delta^3\big(\mathbf{x}-\mathbf{x}_n(t)\big)\Big]. \tag{6.17}$$

进一步，我们有

$$\begin{aligned}\partial_\mu T^{\mu\nu}&=\sum_n m_n\int d\tau_n\big(\frac{dx_n^\mu}{d\tau_n}\frac{dx_n^\nu}{d\tau_n}\big)\partial_\mu\delta^4\big(x-x_n(\tau_n)\big).\\&=-\Big[\sum_n m_n\int d\tau_n\frac{dx_n^\nu}{d\tau_n}\frac{dx_n^\mu}{d\tau_n}\frac{\partial}{\partial x_n^\mu}\delta^4\big(x-x_n(\tau_n)\big)\Big]\\&=-\Big[\sum_n\int d\tau_n p_n^\nu\frac{d}{d\tau_n}\delta^4\big(x-x_n(\tau_n)\big)\Big].\end{aligned} \tag{6.18}$$

将上式分部积分，并丢弃$\tau_n\to\infty$的边界项[1]，即可以得到

$$\partial_\mu T^{\mu\nu}=\sum_n\int d\tau_n\frac{dp_n^\nu}{d\tau_n}\delta^4\big(x-x_n(\tau_n)\big). \tag{6.19}$$

对于我们所考察的自由粒子系统，$\frac{dp_n^\mu}{d\tau_n}=0$，从而上式就给出自由粒子系统的能动量张量守恒方程。但是，即使对于相互作用的粒子系统，(6.19)式依然是成立的，只不过，这时候方程(6.19)的右边将取决于粒子间的相互作用力场。

[1]边界项$-\Big[\sum_n p_n^\nu(\pm\infty)\delta^4\big(x-x_n(\pm\infty)\big)\Big]$之所以可以丢弃，是因为$x_n(\pm\infty)$位于时空的无穷远过去和无穷远将来，而$x$点位于有限区域内，从而$\delta^4\big(x-x_n(\pm\infty)\big)=0$，即边界项其实等于零。

6.2　耦合到电磁场

现在，假设前面考察的粒子是带电粒子，因此可以耦合到电磁场。根据《经典力学新讲》第三章的知识可知，与电磁场耦合的带电粒子系统的作用量为

$$S[x(s)] = -\sum_n m_n \int d\tau_n + \sum_n q_n \int A_\mu dx_n^\mu. \tag{6.20}$$

式中q_n为粒子n的电荷。为了得到系统的运动微分方程，我们将这个作用量对粒子坐标变分，其中第一项的变分与(6.2)式完全相同，关键是要将第二项，即与规范势A_μ耦合的项进行变分，则有

$$\begin{aligned}\sum_n q_n \delta \int A_\mu dx_n^\mu &= \sum_n q_n \int \delta A_\nu dx_n^\nu + \sum_n q_n \int A_\mu d(\delta x_n^\mu)\\ &= \sum_n q_n \int \big(\partial_\mu A_\nu \delta x_n^\mu dx_n^\nu - dA_\mu \delta x_n^\mu\big)\\ &= \sum_n q_n \int \big(\partial_\mu A_\nu \delta x_n^\mu dx_n^\nu - \partial_\nu A_\mu dx_n^\nu \delta x_n^\mu\big)\\ &= \sum_n q_n \int d\tau_n F_{\mu\nu} \frac{dx_n^\nu}{d\tau_n} \delta x_n^\mu. \end{aligned} \tag{6.21}$$

其中第二个等号进行了分部积分，并注意到边界项等于零。从而即可得到与电磁场耦合的作用量(6.20)的变分，

$$\delta S[x(s)] = \sum_n \int d\tau_n \Big[- m_n \frac{d^2 x_n^\mu}{d\tau_n^2} + q_n F^\mu{}_\nu \frac{dx_n^\nu}{d\tau_n}\Big] \delta x_{n\mu}. \tag{6.22}$$

根据最小作用量原理，即可得粒子的运动微分方程，为

$$\frac{dp_n^\mu}{d\tau_n} = m_n \frac{d^2 x_n^\mu}{d\tau_n^2} = q_n F^\mu{}_\nu \frac{dx_n^\nu}{d\tau_n}. \tag{6.23}$$

很显然，方程(6.23)也可以写作

$$\frac{dp_n^\mu}{dt} = q_n F^\mu{}_\nu \frac{dx_n^\nu}{dt}. \tag{6.24}$$

从这里读者不难得到

$$\frac{d\mathbf{p}_n}{dt} = q_n(\mathbf{E} + \mathbf{v}_n \times \mathbf{B}). \tag{6.25}$$

这正是通常所谓的洛伦兹力公式。

将作用量(6.20)对规范势A_μ变分，即可得到

$$\begin{aligned}\delta S &= \sum_n q_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n}\delta A_\mu \\ &= \int d^4x \big[\sum_n q_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n}\delta^4\big(x - x_n(\tau_n)\big)\big]\delta A_\mu. \end{aligned} \tag{6.26}$$

将这个式子与标准的电流四矢量J^μ与规范势的耦合$\delta S = \int d^4x J^\mu \delta A_\mu$进行比较，即可得到此带电粒子系统的电流四矢量，为

$$\begin{aligned}J^\mu(x) &= \sum_n q_n \int d\tau_n \frac{dx_n^\mu}{d\tau_n}\delta^4\big(x - x_n(\tau_n)\big) \\ &= \sum_n q_n \int ds_n \frac{dx_n^\mu}{ds_n}\delta^4\big(x - x_n(s_n)\big). \end{aligned} \tag{6.27}$$

为了看清楚(6.27)式的物理意义，我们取世界线参数$s_n = x_n^0$, 并做出(6.27)式中的积分，则有

$$\begin{aligned}\rho &= \sum_n q_n \delta^3(\mathbf{x} - \mathbf{x}_n(t)) \\ \mathbf{J} &= \sum_n q_n \mathbf{v}_n \delta^3(\mathbf{x} - \mathbf{x}_n(t)). \end{aligned} \tag{6.28}$$

很显然，结果正符合我们对电荷密度以及电流密度表达式的预期。

下面回到多粒子系统的能动量张量。当我们将(6.23)式代入(6.19)式，即有(对与电磁场耦合的带电粒子系统)，

$$\partial_\mu T^{\mu\nu} = \sum_n \int d\tau_n q_n F^\nu{}_\rho \frac{dx_n^\rho}{d\tau_n}\delta^4\big(x - x_n(\tau_n)\big). \tag{6.29}$$

注意到电流四矢量的定义式(6.27)，即有

$$\partial_\mu T^{\mu\nu} = F^\nu{}_\rho J^\rho(x). \tag{6.30}$$

现在，假设电磁场本身就是这些带电粒子产生的，因此要将它包括进我们的系统之中。从而整个带电粒子与电磁场耦合系统的作用量为，

$$S[x(s), A_\mu] = -\int d^4x \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \sum_n m_n \int d\tau_n + \sum_n q_n \int A_\mu dx_n^\mu. \tag{6.31}$$

电磁场当然也会对系统的能动量张量有贡献，根据第四章的知识，电磁场贡献的能动量张量为

$$T^{\mu\nu}_{\rm em} = F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}. \tag{6.32}$$

由于要和带电粒子耦合，$T^{\mu\nu}_{\rm em}$本身当然也是不守恒的，它满足

$$\begin{aligned}\partial_\mu T^{\mu\nu}_{\rm em} &= \partial_\mu F^{\mu}{}_{\rho}F^{\nu\rho} + F_{\mu\rho}\partial^\mu F^{\nu\rho} - \frac{1}{4}\partial^\nu\big(F_{\mu\rho}F^{\mu\rho}\big)\\ &= \partial_\mu F^{\mu}{}_{\rho}F^{\nu\rho} + \frac{1}{2}F_{\mu\rho}\big(\partial^\mu F^{\nu\rho} - \partial^\rho F^{\nu\mu}\big) - \frac{1}{2}F_{\mu\rho}\partial^\nu F^{\mu\rho}\\ &= \partial_\mu F^{\mu}{}_{\rho}F^{\nu\rho} + \frac{1}{2}F_{\mu\rho}\big(\partial^\mu F^{\nu\rho} + \partial^\nu F^{\rho\mu} + \partial^\rho F^{\mu\nu}\big).\end{aligned} \tag{6.33}$$

代入麦克斯韦方程组，即得

$$\partial_\mu T^{\mu\nu}_{\rm em} = -F^{\nu}{}_{\rho}J^{\rho}. \tag{6.34}$$

联合前面的(6.30)式，即可知如下总的能动量张量是守恒的,

$$T^{\mu\nu} = F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + \sum_n m_n\int d\tau_n\Big(\frac{dx^\mu_n}{d\tau_n}\frac{dx^\nu_n}{d\tau_n}\Big)\delta^4\big(x - x_n(\tau_n)\big). \tag{6.35}$$

满足

$$\partial_\mu T^{\mu\nu} = 0. \tag{6.36}$$

另外，注意到$T^{\mu}{}_{\mu\ \rm em} = 0$, 以及$\eta_{\mu\nu}\frac{dx^\mu_n}{d\tau_n}\frac{dx^\nu_n}{d\tau_n} = -1$ (即$-d\tau_n^2 = \eta_{\mu\nu}dx^\mu_n dx^\nu_n$), 即可知这个总能量动量张量满足如下特性

$$T^{\mu}{}_{\mu} = -\sum_n m_n\int d\tau_n\delta^4\big(x - x_n(\tau_n)\big) \le 0. \tag{6.37}$$